

SOME GENERALIZATIONS OF INTEGRAL INEQUALITIES AND THEIR APPLICATIONS

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ABSTRACT. In this paper, an integral identity for twice differentiable functions is generalized. Then, by using convexity of $|f''|$ or $|f''|^q$ and with the aid of power mean and Hölder's inequalities we achieved some new results. We also gave some applications to quadrature formulas and some special means. Therewithal, by choosing $\alpha = \frac{1}{2}$ in our main results, we obtained some findings in [13].

1. INTRODUCTION

We shall recall the definitions of convex functions:

Let I be an interval in \mathbb{R} . Then $f : I \rightarrow \mathbb{R}$ is said to be convex if for all $x, y \in I$ and all $\alpha \in [0, 1]$,

$$(1.1) \quad f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

holds. If (1.1) is strict for all $x \neq y$ and $\alpha \in (0, 1)$, then f is said to be strictly convex. If the inequality in (1.1) is reversed, then f is said to be concave. If it is strict for all $x \neq y$ and $\alpha \in (0, 1)$, then f is said to be strictly concave.

The following inequality is called Hermite-Hadamard inequality for convex functions:

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$(1.2) \quad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}$$

is known in the literature as Hadamard inequality for convex mapping. Note that some of the classical inequalities for means can be derived from (1.2) for appropriate particular selections of the mapping f . Both inequalities hold in the reversed direction if f is concave.

It is well known that the Hermite-Hadamard's inequality plays an important role in nonlinear analysis. Over the last decade, this classical inequality has been improved and generalized in a number of ways; there have been a large number of research papers written on this subject, (see, [1]-[12]) and the references therein.

In [13], Sarikaya *et. al.* established inequalities for twice differentiable convex mappings which are connected with Hadamard's inequality. They used the following lemma and proved next two theorems:

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Lemma 1. Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$. If $f : I \rightarrow \mathbb{R}$ is a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. Then the following identity holds:

(1.3)

$$(\lambda-1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b f(x)dx = (b-a)^2 \int_0^1 k(t)f''(ta+(1-t)b)dt$$

where

$$k(t) = \begin{cases} t(t-\lambda)/2, & 0 \leq t \leq 1/2 \\ (1-t)(1-\lambda-t)/2, & 1/2 \leq t \leq 1. \end{cases}$$

Theorem 1. Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \rightarrow \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. If $|f''|$ is a convex on $[a, b]$, then the following inequalities hold:

(1.4)

$$\begin{aligned} & \left| (\lambda-1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \begin{cases} \frac{(b-a)^2}{12} \left[\left(\lambda^4 + (1+\lambda)(1-\lambda)^3 + \frac{5\lambda-3}{4} \right) |f''(a)| \right. \\ \quad \left. + \left(\lambda^4 + (2-\lambda)\lambda^3 + \frac{1-3\lambda}{4} \right) |f''(b)| \right], & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{(b-a)^2(3\lambda-1)}{48} [|f''(a)| + |f''(b)|] & \text{for } \frac{1}{2} \leq \lambda \leq 1. \end{cases} \end{aligned}$$

Theorem 2. Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : I \rightarrow \mathbb{R}$ be a twice differentiable mapping such that f'' is integrable and $0 \leq \lambda \leq 1$. If $|f''|^q$ is a convex on $[a, b]$, $q \geq 1$, then the following inequalities hold:

(1.5)

$$\begin{aligned} & \left| (\lambda-1)f\left(\frac{a+b}{2}\right) - \lambda \frac{f(a)+f(b)}{2} + \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \begin{cases} \frac{(b-a)^2}{2} \left(\frac{\lambda^3}{3} + \frac{1-3\lambda}{24} \right)^{1-\frac{1}{q}} \\ \quad \times \left\{ \left(\left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3 \times 2^6} \right] |f''(a)|^q + \left[\frac{(2-\lambda)\lambda^3}{6} + \frac{5-16\lambda}{3 \times 2^6} \right] |f''(b)|^q \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\left[\frac{1+\lambda}{6}(1-\lambda)^3 + \frac{48\lambda-27}{3 \times 2^6} \right] |f''(a)|^q + \left[\frac{\lambda^4}{6} + \frac{3-8\lambda}{3 \times 2^6} \right] |f''(b)|^q \right)^{\frac{1}{q}} \right\}, & \text{for } 0 \leq \lambda \leq \frac{1}{2} \\ \frac{(b-a)^2}{2} \left(\frac{3\lambda-1}{24} \right)^{1-\frac{1}{q}} \left\{ \left(\frac{8\lambda-3}{3 \times 2^6} |f''(a)|^q + \frac{16\lambda-5}{3 \times 2^6} |f''(b)|^q \right)^{\frac{1}{q}} \right. \\ \quad \left. + \left(\frac{16\lambda-5}{3 \times 2^6} |f''(a)|^q + \frac{8\lambda-3}{3 \times 2^6} |f''(b)|^q \right)^{\frac{1}{q}} \right\} & \text{for } \frac{1}{2} \leq \lambda \leq 1, \end{cases} \end{aligned}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

For some recent results connected with twice differentiable functions, see ([13]-[16]).

In this paper, we achieved an integral identity for twice differentiable functions. Then, by using convexity of $|f''|$ or $|f''|^q$ we achieved some new results. We also gave some applications to quadrature formulas and some special means.

2. MAIN RESULTS

In order to prove our theorems we need following Lemma:

Lemma 2. *Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f, f', f'' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. Then the following equality holds:*

$$\begin{aligned}
 (2.1) \quad & (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \\
 & + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda(\alpha f(a) + (1-\alpha)f(b)) \\
 & = \frac{(b-a)^2}{2} \int_0^1 k(t) f''(tb + (1-t)a) dt
 \end{aligned}$$

where

$$k(t) = \begin{cases} 2\alpha\lambda t - t^2, & 0 \leq t \leq 1-\alpha \\ (1-t)(t-1+2\lambda(1-\alpha)), & 1-\alpha \leq t \leq 1. \end{cases}$$

Proof. We note that

$$\begin{aligned}
 I &= \int_0^{1-\alpha} (2\alpha\lambda t - t^2) f''(tb + (1-t)a) dt \\
 &+ \int_{1-\alpha}^1 (1-t)(t-1+2\lambda(1-\alpha)) f''(tb + (1-t)a) dt.
 \end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}
 I &= (2\alpha\lambda t - t^2) \frac{f'(tb + (1-t)a)}{b-a} \Big|_0^{1-\alpha} \\
 &- \int_0^{1-\alpha} (2\alpha\lambda - 2t) \frac{f'(tb + (1-t)a)}{b-a} dt \\
 &+ \left(\frac{(1-t)(t-1+2\lambda(1-\alpha)) f'(tb + (1-t)a)}{(b-a)} \right) \Big|_{1-\alpha}^1 \\
 &- \int_{1-\alpha}^1 2(1-t-\lambda(1-\alpha)) \frac{f'(tb + (1-t)a)}{b-a} dt \\
 &= \left(2\alpha\lambda(1-\alpha) - (1-\alpha)^2 \right) \frac{f'((1-\alpha)b + \alpha a)}{b-a} \\
 &- (\alpha(-\alpha + 2\lambda(1-\alpha))) \frac{f'((1-\alpha)b + \alpha a)}{b-a} \\
 &- \frac{2}{b-a} \left\{ \int_0^{1-\alpha} (\alpha\lambda - t) f'(tb + (1-t)a) dt \right. \\
 &\quad \left. + \int_{1-\alpha}^1 (1-t-\lambda(1-\alpha)) f'(tb + (1-t)a) dt \right\}.
 \end{aligned}$$

By simple calculation we have

$$\begin{aligned}
I &= (2\alpha - 1) \frac{f'((1-\alpha)b + \alpha a)}{b-a} \\
&\quad - \frac{2}{b-a} \left\{ (\alpha\lambda - t) \frac{f(tb+(1-t)a)}{b-a} \Big|_0^{1-\alpha} - \int_0^{1-\alpha} \frac{-f(tb+(1-t)a)}{b-a} dt \right. \\
&\quad \left. + (1-t-\lambda(1-\alpha)) \frac{f(tb+(1-t)a)}{b-a} \Big|_{1-\alpha}^1 - \int_{1-\alpha}^1 \frac{-f(tb+(1-t)a)}{b-a} dt \right\} \\
&= (2\alpha - 1) \frac{f'((1-\alpha)b + \alpha a)}{b-a} \\
&\quad - \frac{2}{b-a} \left\{ (\alpha\lambda - 1 + \alpha) \frac{f((1-\alpha)b + \alpha a)}{b-a} - \frac{\alpha\lambda f(a)}{b-a} + \left(\frac{-\lambda(1-\alpha)f(b)}{b-a} \right) \right. \\
&\quad \left. - (\alpha - \lambda(1-\alpha)) \frac{f((1-\alpha)b + \alpha a)}{b-a} + \int_0^1 \frac{f(tb+(1-t)a)}{b-a} dt \right\} \\
&= (2\alpha - 1) \frac{f'((1-\alpha)b + \alpha a)}{b-a} \\
&\quad + \frac{2}{(b-a)^2} \left\{ (1-\lambda) f((1-\alpha)b + \alpha a) \right. \\
&\quad \left. + \lambda(\alpha f(a) + (1-\alpha)f(b)) - \int_0^1 f(tb+(1-t)a) dt \right\}.
\end{aligned}$$

By change of variable $x = tb + (1-t)a$ and multiplying both sides with $\frac{(b-a)^2}{2}$ we get the desired result. \square

Theorem 3. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f, f', f'' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f''|$ is convex on $[a, b]$, then the following inequalities hold:

$$\begin{aligned}
(2.2) \quad & \left| (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \right. \\
& \left. + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda(\alpha f(a) + (1-\alpha)f(b)) \right| \\
& \leq \begin{cases} \frac{(b-a)^2}{2} \{(\gamma_1 + \mu_1) |f''(b)| + (\gamma_2 + \mu_2) |f''(a)|\}, & 2\alpha\lambda \leq 1-\alpha \leq 1-2\lambda(1-\alpha) \\ \frac{(b-a)^2}{2} \{(\gamma_1 + \mu_3) |f''(b)| + (\gamma_2 + \mu_4) |f''(a)|\}, & 1-\alpha \geq \max\{2\alpha\lambda, 1-2\lambda(1-\alpha)\} \\ \frac{(b-a)^2}{2} \{(\gamma_3 + \mu_1) |f''(b)| + (\gamma_4 + \mu_2) |f''(a)|\}, & 1-\alpha \leq \min\{2\alpha\lambda, 1-2\lambda(1-\alpha)\} \\ \frac{(b-a)^2}{2} \{(\gamma_3 + \mu_3) |f''(b)| + (\gamma_4 + \mu_4) |f''(a)|\}, & 1-2\lambda(1-\alpha) \leq 1-\alpha \leq 2\alpha\lambda, \end{cases}
\end{aligned}$$

where

$$\begin{aligned}
\gamma_1 &= \frac{8}{3} (\alpha\lambda)^4 + (1-\alpha)^3 \left(\frac{1-\alpha}{4} - \frac{2\alpha\lambda}{3} \right) \\
\gamma_2 &= \frac{8}{3} (\alpha\lambda)^3 (1-\alpha\lambda) + (1-\alpha)^2 \left[\frac{1-\alpha}{3} - \alpha\lambda - \frac{(1-\alpha)^2}{4} + \frac{2\alpha\lambda(1-\alpha)}{3} \right] \\
\gamma_3 &= \frac{2\alpha\lambda(1-\alpha)^3}{3} - \frac{(1-\alpha)^4}{4} \\
\gamma_4 &= (1-\alpha)^2 \left[\alpha\lambda - \frac{1-\alpha}{3} - \frac{2\alpha\lambda(1-\alpha)}{3} + \frac{(1-\alpha)^2}{4} \right]
\end{aligned}$$

and

$$\begin{aligned}
 \mu_1 &= \frac{4}{3} (1-\alpha)^3 \lambda^3 (1-\lambda(1-\alpha)) - \frac{1}{12} (\alpha - 2\lambda(1-\alpha))^2 [\alpha(3\alpha-4) - 4\lambda(1-\alpha)^2(1-\lambda)] \\
 \mu_2 &= \frac{4}{3} (1-\alpha)^4 \lambda^4 + \frac{1}{12} (\alpha - 2\lambda(1-\alpha))^2 [\alpha^2(4\lambda^4 - 4\lambda + 3) + 4\alpha\lambda(1-2\lambda) + 4\lambda^2] \\
 \mu_3 &= \frac{4}{3} (1-\alpha)^3 \lambda^3 (1-\lambda(1-\alpha)) + \frac{1}{12} (\alpha - 2\lambda(1-\alpha))^2 [\alpha(3\alpha-4) - 4\lambda(1-\alpha)^2(1-\lambda)] \\
 \mu_4 &= \frac{4}{3} (1-\alpha)^4 \lambda^4 - \frac{1}{12} (\alpha - 2\lambda(1-\alpha))^2 [\alpha^2(4\lambda^4 - 4\lambda + 3) + 4\alpha\lambda(1-2\lambda) + 4\lambda^2].
 \end{aligned}$$

Proof. By using Lemma 2, properties of absolute value and using convexity of $|f''|$ we have,

$$\begin{aligned}
 & \left| (b-a) \left(\alpha - \frac{1}{2}\right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \left| + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda(\alpha f(a) + (1-\alpha)f(b)) \right| \\
 & \leq \frac{(b-a)^2}{2} \left\{ \int_0^{1-\alpha} t |2\alpha\lambda - t| |f''(tb + (1-t)a)| dt \right. \\
 & \quad \left. + \int_{1-\alpha}^1 (1-t) |1 - 2\lambda(1-\alpha) - t| |f''(tb + (1-t)a)| dt \right\} \\
 & \leq \frac{(b-a)^2}{2} \left\{ \int_0^{1-\alpha} t |2\alpha\lambda - t| (t |f''(b)| + (1-t) |f''(a)|) dt \right. \\
 & \quad \left. + \int_{1-\alpha}^1 (1-t) |1 - 2\lambda(1-\alpha) - t| (t |f''(b)| + (1-t) |f''(a)|) dt \right\} \\
 & = \frac{(b-a)^2}{2} \left\{ \int_0^{1-\alpha} |2\alpha\lambda - t| (t^2 |f''(b)| + t(1-t) |f''(a)|) dt \right. \\
 & \quad \left. + \int_{1-\alpha}^1 |1 - 2\lambda(1-\alpha) - t| (t(1-t) |f''(b)| + (1-t)^2 |f''(a)|) dt \right\}. \tag{2.3}
 \end{aligned}$$

Hence by simple calculation

$$\begin{aligned}
 (2.4) \quad & \int_0^{1-\alpha} |2\alpha\lambda - t| (t^2 |f''(b)| + t(1-t) |f''(a)|) dt \\
 & = \begin{cases} \gamma_1 |f''(b)| + \gamma_2 |f''(a)|, & 2\alpha\lambda \leq 1-\alpha \\ \gamma_3 |f''(b)| + \gamma_4 |f''(a)|, & 2\alpha\lambda \geq 1-\alpha \end{cases},
 \end{aligned}$$

$$\begin{aligned}
 \gamma_1 &= \frac{8}{3} (\alpha\lambda)^4 + (1-\alpha)^3 \left(\frac{1-\alpha}{4} - \frac{2\alpha\lambda}{3} \right) \\
 \gamma_2 &= \frac{8}{3} (\alpha\lambda)^3 (1-\alpha\lambda) + (1-\alpha)^2 \left[\frac{1-\alpha}{3} - \alpha\lambda - \frac{(1-\alpha)^2}{4} + \frac{2\alpha\lambda(1-\alpha)}{3} \right] \\
 \gamma_3 &= \frac{2\alpha\lambda(1-\alpha)^3}{3} - \frac{(1-\alpha)^4}{4} \\
 \gamma_4 &= (1-\alpha)^2 \left[\alpha\lambda - \frac{1-\alpha}{3} - \frac{2\alpha\lambda(1-\alpha)}{3} + \frac{(1-\alpha)^2}{4} \right]
 \end{aligned}$$

and

$$\begin{aligned}
 (2.5) \quad & \int_{1-\alpha}^1 |1 - 2\lambda(1-\alpha) - t| \left(t(1-t)|f''(b)| + (1-t)^2|f''(a)| \right) dt \\
 &= \begin{cases} \mu_1 |f''(b)| + \mu_2 |f''(a)|, & 1-\alpha \leq 1-2\lambda(1-\alpha) \\ \mu_3 |f''(b)| + \mu_4 |f''(a)|, & 1-\alpha \geq 1-2\lambda(1-\alpha) \end{cases}, \\
 \mu_1 &= \frac{4}{3}(1-\alpha)^3 \lambda^3 (1-\lambda(1-\alpha)) - \frac{1}{12}(\alpha - 2\lambda(1-\alpha))^2 [\alpha(3\alpha-4) - 4\lambda(1-\alpha)^2(1-\lambda)] \\
 \mu_2 &= \frac{4}{3}(1-\alpha)^4 \lambda^4 + \frac{1}{12}(\alpha - 2\lambda(1-\alpha))^2 [\alpha^2(4\lambda^4 - 4\lambda + 3) + 4\alpha\lambda(1-2\lambda) + 4\lambda^2] \\
 \mu_3 &= \frac{4}{3}(1-\alpha)^3 \lambda^3 (1-\lambda(1-\alpha)) + \frac{1}{12}(\alpha - 2\lambda(1-\alpha))^2 [\alpha(3\alpha-4) - 4\lambda(1-\alpha)^2(1-\lambda)] \\
 \mu_4 &= \frac{4}{3}(1-\alpha)^4 \lambda^4 - \frac{1}{12}(\alpha - 2\lambda(1-\alpha))^2 [\alpha^2(4\lambda^4 - 4\lambda + 3) + 4\alpha\lambda(1-2\lambda) + 4\lambda^2].
 \end{aligned}$$

Thus, using (2.4) and (2.5) in (2.3), we obtain (2.2). This completes the proof. \square

Theorem 4. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f, f', f'' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f''|^q$ is convex on $[a, b]$, for $q \geq 1$, the following inequalities hold:

$$\begin{aligned}
 (2.6) \quad & \left| \begin{aligned} & (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \\ & + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda(\alpha f(a) + (1-\alpha)f(b)) \end{aligned} \right| \\
 & \leq \begin{cases} \frac{(b-a)^2}{2} \left[\begin{aligned} & \tau_1^{1-\frac{1}{q}} (\gamma_1 |f''(b)|^q + \gamma_2 |f''(a)|^q)^{\frac{1}{q}} \\ & + z_1^{1-\frac{1}{q}} (\mu_1 |f''(b)|^q + \mu_2 |f''(a)|^q)^{\frac{1}{q}} \end{aligned} \right], & 2\alpha\lambda \leq 1-\alpha \leq 1-2\lambda(1-\alpha) \\ \frac{(b-a)^2}{2} \left[\begin{aligned} & \tau_1^{1-\frac{1}{q}} (\gamma_1 |f''(b)|^q + \gamma_2 |f''(a)|^q)^{\frac{1}{q}} \\ & + z_2^{1-\frac{1}{q}} (\mu_3 |f''(b)|^q + \mu_4 |f''(a)|^q)^{\frac{1}{q}} \end{aligned} \right], & 1-\alpha \geq \max\{2\alpha\lambda, 1-2\lambda(1-\alpha)\} \\ \frac{(b-a)^2}{2} \left[\begin{aligned} & \tau_2^{1-\frac{1}{q}} (\gamma_3 |f''(b)|^q + \gamma_4 |f''(a)|^q)^{\frac{1}{q}} \\ & + z_1^{1-\frac{1}{q}} (\mu_1 |f''(b)|^q + \mu_2 |f''(a)|^q)^{\frac{1}{q}} \end{aligned} \right], & 1-\alpha \leq \min\{2\alpha\lambda, 1-2\lambda(1-\alpha)\} \\ \frac{(b-a)^2}{2} \left[\begin{aligned} & \tau_2^{1-\frac{1}{q}} (\gamma_3 |f''(b)|^q + \gamma_4 |f''(a)|^q)^{\frac{1}{q}} \\ & + z_2^{1-\frac{1}{q}} (\mu_3 |f''(b)|^q + \mu_4 |f''(a)|^q)^{\frac{1}{q}} \end{aligned} \right], & 1-2\lambda(1-\alpha) \leq 1-\alpha \leq 2\alpha\lambda, \end{cases}
 \end{aligned}$$

where

$$\begin{aligned}
 \tau_1 &= \frac{8}{3}(\alpha\lambda)^3 + (1-\alpha)^2 \left(\frac{1-\alpha}{3} - \alpha\lambda \right) \\
 \tau_2 &= (1-\alpha)^2 \left(\alpha\lambda - \frac{1-\alpha}{3} \right)
 \end{aligned}$$

and

$$\begin{aligned}
 z_1 &= \frac{4}{3}(1-\alpha)^3 \lambda^3 + \frac{1}{3}(\alpha - 2\lambda(1-\alpha))^2 (\alpha(1-\lambda) + \lambda) \\
 z_2 &= \frac{4}{3}(1-\alpha)^3 \lambda^3 - \frac{1}{3}(\alpha - 2\lambda(1-\alpha))^2 (\alpha(1-\lambda) + \lambda)
 \end{aligned}$$

γ_i and μ_i ($i = 1, 2, 3, 4$) are defined as in Theorem 3.

Proof. Suppose that $q \geq 1$. From Lemma 2 and using well known power mean inequality, we have

$$\begin{aligned}
 & \left| (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \quad + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda (\alpha f(a) + (1-\alpha) f(b)) \Big| \\
 & \leq \frac{(b-a)^2}{2} \left\{ \int_0^{1-\alpha} |2\alpha\lambda t - t^2| |f''(tb + (1-t)a)| dt \right. \\
 & \quad \left. + \int_{1-\alpha}^1 |(1-t)(1-2\lambda(1-\alpha)-t)| |f''(tb + (1-t)a)| dt \right\} \\
 & \leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1-\alpha} |2\alpha\lambda t - t^2| dt \right)^{1-\frac{1}{q}} \right. \\
 & \quad \times \left(\int_0^{1-\alpha} |2\alpha\lambda t - t^2| |f''(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \\
 & \quad + \left(\int_{1-\alpha}^1 |(1-t)(1-2\lambda(1-\alpha)-t)| dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left. \left(\int_{1-\alpha}^1 |(1-t)(1-2\lambda(1-\alpha)-t)| |f''(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

By using convexity of $|f''|^q$ we know that

$$|f''(ta + (1-t)b)|^q \leq t |f''(a)|^q + (1-t) |f''(b)|^q.$$

So we have

$$\begin{aligned}
 & \left| (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \quad + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda (\alpha f(a) + (1-\alpha) f(b)) \Big| \\
 & \leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1-\alpha} t |2\alpha\lambda - t| dt \right)^{1-\frac{1}{q}} \right. \\
 & \quad \times \left(\int_0^{1-\alpha} |2\alpha\lambda - t| (t^2 |f''(b)|^q + t(1-t) |f''(a)|^q) dt \right)^{\frac{1}{q}} \\
 & \quad (2.7) \quad + \left(\int_{1-\alpha}^1 (1-t) |(1-2\lambda(1-\alpha)-t)| dt \right)^{1-\frac{1}{q}} \\
 & \quad \times \left. \left(\int_{1-\alpha}^1 |(1-2\lambda(1-\alpha)-t)| (t(1-t) |f''(b)|^q + (1-t)^2 |f''(a)|^q) dt \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

Hence, by simple computation

$$(2.8) \quad \int_0^{1-\alpha} t |2\alpha\lambda - t| dt = \begin{cases} \tau_1, & 2\alpha\lambda \leq 1-\alpha \\ \tau_2, & 2\alpha\lambda \geq 1-\alpha \end{cases},$$

$$\begin{aligned}\tau_1 &= \frac{8}{3}(\alpha\lambda)^3 + (1-\alpha)^2 \left(\frac{1-\alpha}{3} - \alpha\lambda \right) \\ \tau_2 &= (1-\alpha)^2 \left(\alpha\lambda - \frac{1-\alpha}{3} \right)\end{aligned}$$

$$(2.9) \quad \int_{1-\alpha}^1 (1-t) |(1-2\lambda(1-\alpha)-t)| dt = \begin{cases} z_1, & 1-2\lambda(1-\alpha) \geq 1-\alpha \\ z_2, & 1-2\lambda(1-\alpha) \leq 1-\alpha \end{cases}$$

$$\begin{aligned}z_1 &= \frac{4}{3}(1-\alpha)^3 \lambda^3 + \frac{1}{3}(\alpha-2\lambda(1-\alpha))^2 (\alpha(1-\lambda) + \lambda) \\ z_2 &= \frac{4}{3}(1-\alpha)^3 \lambda^3 - \frac{1}{3}(\alpha-2\lambda(1-\alpha))^2 (\alpha(1-\lambda) + \lambda)\end{aligned}$$

$$\begin{aligned}(2.10) \quad & \int_0^{1-\alpha} |2\alpha\lambda - t| (t^2 |f''(b)|^q + t(1-t) |f''(a)|^q) dt \\ &= \begin{cases} \gamma_1 |f''(b)|^q + \gamma_2 |f''(a)|^q, & 2\alpha\lambda \leq 1-\alpha \\ \gamma_3 |f''(b)|^q + \gamma_4 |f''(a)|^q, & 2\alpha\lambda \geq 1-\alpha \end{cases}\end{aligned}$$

and $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ are defined as in Theorem 3.

$$\begin{aligned}(2.11) \quad & \int_{1-\alpha}^1 |(1-2\lambda(1-\alpha)-t)| (t(1-t) |f''(b)|^q + (1-t)^2 |f''(a)|^q) dt \\ &= \begin{cases} \mu_1 |f''(b)|^q + \mu_2 |f''(a)|^q, & 1-\alpha \leq 1-2\lambda(1-\alpha) \\ \mu_3 |f''(b)|^q + \mu_4 |f''(a)|^q, & 1-\alpha \geq 1-2\lambda(1-\alpha) \end{cases}\end{aligned}$$

and $\mu_1, \mu_2, \mu_3, \mu_4$ are defined as in Theorem 3. Thus, using (2.8)-(2.11) in (2.7), we get (2.6). So the proof is completed. \square

Theorem 5. Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable mapping on I° such that $f, f', f'' \in L[a, b]$, where $a, b \in I$ with $a < b$ and $\alpha, \lambda \in [0, 1]$. If $|f''|^q$ is convex on

$[a, b]$, for $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$, the following inequalities hold:

$$\begin{aligned} & \left| (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda (\alpha f(a) + (1-\alpha) f(b)) \\ & \leq \begin{cases} \frac{(b-a)^2}{2} \left\{ \varphi_1^{\frac{1}{p}} \left[\varepsilon_1 |f''(b)|^q + \beta(1-\alpha; q+1, 2) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ \quad \left. + \psi_1^{\frac{1}{p}} \left[\beta(\alpha; q+1, 2) |f''(b)|^q + \varepsilon_2 |f''(a)|^q \right]^{\frac{1}{q}} \right\}, & 2\alpha\lambda \leq 1-\alpha \leq 1-2\lambda(1-\alpha) \\ \frac{(b-a)^2}{2} \left\{ \varphi_1^{\frac{1}{p}} \left[\varepsilon_1 |f''(b)|^q + \beta(1-\alpha; q+1, 2) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ \quad \left. + \psi_2^{\frac{1}{p}} \left[\beta(\alpha; q+1, 2) |f''(b)|^q + \varepsilon_2 |f''(a)|^q \right]^{\frac{1}{q}} \right\}, & 1-\alpha \geq \max\{2\alpha\lambda, 1-2\lambda(1-\alpha)\} \\ \frac{(b-a)^2}{2} \left\{ \varphi_2^{\frac{1}{p}} \left[\varepsilon_1 |f''(b)|^q + \beta(1-\alpha; q+1, 2) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ \quad \left. + \psi_1^{\frac{1}{p}} \left[\beta(\alpha; q+1, 2) |f''(b)|^q + \varepsilon_2 |f''(a)|^q \right]^{\frac{1}{q}} \right\}, & 1-\alpha \leq \min\{2\alpha\lambda, 1-2\lambda(1-\alpha)\} \\ \frac{(b-a)^2}{2} \left\{ \varphi_2^{\frac{1}{p}} \left[\varepsilon_1 |f''(b)|^q + \beta(1-\alpha; q+1, 2) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ \quad \left. + \psi_2^{\frac{1}{p}} \left[\beta(\alpha; q+1, 2) |f''(b)|^q + \varepsilon_2 |f''(a)|^q \right]^{\frac{1}{q}} \right\}, & 1-2\lambda(1-\alpha) \leq 1-\alpha \leq 2\alpha\lambda, \end{cases} \end{aligned}$$

where

$$\begin{aligned} \varphi_1 &= \frac{(2\alpha\lambda)^{p+1} + (1-\alpha(1+2\lambda))^{p+1}}{p+1} \\ \varphi_2 &= \frac{(2\alpha\lambda)^{p+1} - (-1)^{p+1}(1-\alpha(1+2\lambda))^{p+1}}{p+1} \end{aligned}$$

$$\begin{aligned} \psi_1 &= \frac{(2\lambda(1-\alpha))^{p+1} + (\alpha - 2\lambda(1-\alpha))^{p+1}}{p+1} \\ \psi_2 &= \frac{(2\lambda(1-\alpha))^{p+1} - (-1)^{p+1}(\alpha - 2\lambda(1-\alpha))^{p+1}}{p+1} \end{aligned}$$

$$\begin{aligned} \varepsilon_1 &= \frac{(1-\alpha)^{q+2}}{q+2} \\ \varepsilon_2 &= \frac{\alpha^{q+2}}{q+2} \end{aligned}$$

and β is incomplete Beta function.

Proof. From Lemma 2 and properties of absolute value and using Hölder inequality, for $p, q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ we have

$$\begin{aligned}
I &= \left| (b-a) \left(\alpha - \frac{1}{2} \right) f'((1-\alpha)b + \alpha a) - \frac{1}{b-a} \int_a^b f(x) dx \right. \\
&\quad \left. + (1-\lambda) f((1-\alpha)b + \alpha a) + \lambda (\alpha f(a) + (1-\alpha) f(b)) \right| \\
&\leq \frac{(b-a)^2}{2} \left\{ \int_0^{1-\alpha} |2\alpha\lambda t - t^2| |f''(tb + (1-t)a)| dt \right. \\
&\quad \left. + \int_{1-\alpha}^1 |(1-t)(1-2\lambda(1-\alpha)-t)| |f''(tb + (1-t)a)| dt \right\} \\
&= \frac{(b-a)^2}{2} \left[\left(\int_0^{1-\alpha} |2\alpha\lambda - t| [t |f''(tb + (1-t)a)|] dt \right) \right. \\
&\quad \left. + \left(\int_{1-\alpha}^1 |(1-2\lambda(1-\alpha)-t)| [(1-t) |f''(tb + (1-t)a)|] dt \right) \right] \\
&\leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1-\alpha} |2\alpha\lambda - t|^p dt \right)^{\frac{1}{p}} \left(\int_0^{1-\alpha} t^q |f''(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right. \\
&\quad \left. + \left(\int_{1-\alpha}^1 |(1-2\lambda(1-\alpha)-t)|^p dt \right)^{\frac{1}{p}} \left(\int_{1-\alpha}^1 (1-t)^q |f''(tb + (1-t)a)|^q dt \right)^{\frac{1}{q}} \right]
\end{aligned}$$

Since $|f''|^q$ is convex on $[a, b]$ we can write

$$\begin{aligned}
I &\leq \frac{(b-a)^2}{2} \left[\left(\int_0^{1-\alpha} |2\alpha\lambda - t|^p dt \right)^{\frac{1}{p}} \right. \\
&\quad \left(\int_0^{1-\alpha} t^q (t |f''(b)|^q + (1-t) |f''(a)|^q) dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{1-\alpha}^1 |(1-2\lambda(1-\alpha)-t)|^p dt \right)^{\frac{1}{p}} \\
&\quad \left. \left(\int_{1-\alpha}^1 (1-t)^q (t |f''(b)|^q + (1-t) |f''(a)|^q) dt \right)^{\frac{1}{q}} \right] \\
&= \frac{(b-a)^2}{2} \left[\left(\int_0^{1-\alpha} |2\alpha\lambda - t|^p dt \right)^{\frac{1}{p}} \right. \\
&\quad \left(|f''(b)|^q \int_0^{1-\alpha} t^{q+1} dt + |f''(a)|^q \int_0^{1-\alpha} t^q (1-t) dt \right)^{\frac{1}{q}} \\
&\quad + \left(\int_{1-\alpha}^1 |(1-2\lambda(1-\alpha)-t)|^p dt \right)^{\frac{1}{p}} \\
&\quad \left. \left(|f''(b)|^q \int_{1-\alpha}^1 t(1-t)^q dt + |f''(a)|^q \int_{1-\alpha}^1 (1-t)^{q+1} dt \right)^{\frac{1}{q}} \right]. \tag{2.12}
\end{aligned}$$

Making use of necessary computation

$$\int_0^{1-\alpha} |2\alpha\lambda - t|^p dt = \begin{cases} \varphi_1, & 2\alpha\lambda \leq 1-\alpha \\ \varphi_2, & 2\alpha\lambda \geq 1-\alpha \end{cases} \tag{2.13}$$

$$\begin{aligned}
 \varphi_1 &= \frac{(2\alpha\lambda)^{p+1} + (1 - \alpha(1 + 2\lambda))^{p+1}}{p+1} \\
 \varphi_2 &= \frac{(2\alpha\lambda)^{p+1} - (-1)^{p+1}(1 - \alpha(1 + 2\lambda))^{p+1}}{p+1} \\
 (2.14) \quad \int_{1-\alpha}^1 |(1 - 2\lambda(1 - \alpha) - t)|^p dt &= \begin{cases} \psi_1, & 1 - \alpha \leq 1 - 2\lambda(1 - \alpha) \\ \psi_2, & 1 - \alpha \geq 1 - 2\lambda(1 - \alpha) \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 \psi_1 &= \frac{(2\lambda(1 - \alpha))^{p+1} + (\alpha - 2\lambda(1 - \alpha))^{p+1}}{p+1} \\
 \psi_2 &= \frac{(2\lambda(1 - \alpha))^{p+1} - (-1)^{p+1}(\alpha - 2\lambda(1 - \alpha))^{p+1}}{p+1}
 \end{aligned}$$

$$\begin{aligned}
 (2.15) \quad \int_0^{1-\alpha} t^{q+1} dt &= \frac{(1 - \alpha)^{q+2}}{q+2} \\
 \int_{1-\alpha}^1 (1 - t)^{q+1} dt &= \frac{\alpha^{q+2}}{q+2}
 \end{aligned}$$

and

$$\begin{aligned}
 (2.16) \quad \int_0^{1-\alpha} t^q (1 - t) dt &= \beta(1 - \alpha; q + 1, 2) \\
 \int_{1-\alpha}^1 t (1 - t)^q dt &= \beta(\alpha; q + 1, 2).
 \end{aligned}$$

where β is incomplete Beta function. By using (2.13)-(2.16) in (2.12), we get the desired result. \square

Remark 1. In our main results, if we choose $\alpha = \frac{1}{2}$, then under attendant assumptions, Lemma 2, Theorem 3 and Theorem 4 reduces to Lemma 1, Theorem 1 and Theorem 2 in [13], respectively.

Remark 2. Under the assumptions of Lemma 2, by integrating both sides respect to α over $[0, 1]$ we get

$$\begin{aligned}
 &(\lambda - 1) \left(\frac{f(a) + f(b)}{2} - \frac{1}{b - a} \int_a^b f(x) dx \right) \\
 &= \frac{(b - a)^2}{2} \int_0^1 \int_0^1 k(\alpha, t) f''(tb + (1 - t)a) dt d\alpha
 \end{aligned}$$

where

$$k(t) = \begin{cases} 2\alpha\lambda t - t^2, & 0 \leq t \leq 1 - \alpha \\ (1 - t)(t - 1 + 2\lambda(1 - \alpha)), & 1 - \alpha \leq t \leq 1. \end{cases}$$

3. APPLICATIONS TO QUADRATURE FORMULAS

In this section we point out some particular inequalities which generalize some classical results such as : trapezoid inequality, Simpson's inequality, midpoint inequality.

Proposition 1 (Midpoint inequality). *Under the assumptions of Theorem 5 with $\alpha = \frac{1}{2}$, $\lambda = 0$, we get the following inequality:*

$$\begin{aligned} & \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left(\frac{1}{2^{2p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left[\frac{1}{2^{q+2}(q+2)} |f''(b)|^q + \beta\left(\frac{1}{2}; q+1, 2\right) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\beta\left(\frac{1}{2}; q+1, 2\right) |f''(b)|^q + \frac{1}{2^{q+2}(q+2)} |f''(a)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proposition 2 (Trapezoid inequality). *Under the assumptions of Theorem 5 with $\alpha = \frac{1}{2}$, $\lambda = 1$, we get the following inequality:*

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left(\frac{2^{p+1}-1}{2^{2p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left[\frac{1}{2^{q+2}(q+2)} |f''(b)|^q + \beta\left(\frac{1}{2}; q+1, 2\right) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\beta\left(\frac{1}{2}; q+1, 2\right) |f''(b)|^q + \frac{1}{2^{q+2}(q+2)} |f''(a)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proposition 3 (Simpson inequality). *Under the assumptions of Theorem 5 with $\alpha = \frac{1}{2}$, $\lambda = \frac{1}{3}$, we get the following inequality:*

$$\begin{aligned} & \left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq (b-a)^2 \left(\frac{2^{p+1}-1}{2^{2p+1}3^{p+1}(p+1)} \right)^{\frac{1}{p}} \left\{ \left[\frac{1}{2^{q+2}(q+2)} |f''(b)|^q + \beta\left(\frac{1}{2}; q+1, 2\right) |f''(a)|^q \right]^{\frac{1}{q}} \right. \\ & \quad \left. + \left[\beta\left(\frac{1}{2}; q+1, 2\right) |f''(b)|^q + \frac{1}{2^{q+2}(q+2)} |f''(a)|^q \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

4. APPLICATIONS TO SPECIAL MEANS

We now consider some applications with the following special means

a) The arithmetic mean:

$$A = A(a, b) := \frac{a+b}{2}, \quad a, b \geq 0,$$

b) The geometric mean:

$$G = G(a, b) := \sqrt{ab}, \quad a, b \geq 0,$$

c) The harmonic mean:

$$H = H(a, b) := \frac{2ab}{a+b}, \quad a, b > 0,$$

d) The logarithmic mean:

$$L = L(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases}, \quad a, b > 0,$$

e) The identric mean:

$$I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left(\frac{b^b}{a^a} \right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases}, \quad a, b > 0,$$

f) The p -logarithmic mean:

$$L_p = L_p(a, b) := \begin{cases} a & \text{if } a = b \\ \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right]^{\frac{1}{p}} & \text{if } a \neq b \end{cases}, \quad p \in \mathbb{R} \setminus \{-1, 0\}; \quad a, b > 0.$$

We now derive some sophisticated bounds of the above means by using the results at Section 3.

Proposition 4. *Let $a, b \in \mathbb{R}, 0 < a < b$. Then, for all $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned} & |A^{-1}(a, b) - L^{-1}(a, b)| \\ & \leq \frac{(b-a)^2}{8} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)} \right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1} \right)^{i-1} \frac{1}{b^{3q}} + \left(\frac{q+3}{q+1} \right)^{2-i} \frac{1}{a^{3q}} \right]^{\frac{1}{q}} \right\} \\ & \text{and} \\ & |H^{-1}(a, b) - L^{-1}(a, b)| \\ & \leq \frac{(b-a)^2}{8} \left(\frac{2^{p+1}-1}{p+1} \right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)} \right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1} \right)^{i-1} \frac{1}{b^{3q}} + \left(\frac{q+3}{q+1} \right)^{2-i} \frac{1}{a^{3q}} \right]^{\frac{1}{q}} \right\} \\ & \text{and} \\ & \left| \frac{1}{3} H^{-1}(a, b) + \frac{2}{3} A^{-1}(a, b) - L^{-1}(a, b) \right| \\ & \leq \frac{(b-a)^2}{24} \left(\frac{2^{p+1}-1}{3(p+1)} \right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)} \right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1} \right)^{i-1} \frac{1}{b^{3q}} + \left(\frac{q+3}{q+1} \right)^{2-i} \frac{1}{a^{3q}} \right]^{\frac{1}{q}} \right\}. \end{aligned}$$

Proof. The assertions follow from Proposition 1, 2 and 3 applied to convex mapping $f(x) = \frac{1}{x}$, $x \in [a, b]$, respectively. \square

Proposition 5. *Let $a, b \in \mathbb{R}, 0 < a < b$. Then, for all $q \geq 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, we have*

$$\begin{aligned}
& |\ln(A(a, b)) - \ln(I(a, b))| \\
& \leq \frac{(b-a)^2}{16} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)}\right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1}\right)^{i-1} \frac{1}{b^{2q}} + \left(\frac{q+3}{q+1}\right)^{2-i} \frac{1}{a^{2q}} \right]^{\frac{1}{q}} \right\} \\
& \text{and} \\
& |\ln(G(a, b)) - \ln(I(a, b))| \\
& \leq \frac{(b-a)^2}{16} \left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)}\right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1}\right)^{i-1} \frac{1}{b^{2q}} + \left(\frac{q+3}{q+1}\right)^{2-i} \frac{1}{a^{2q}} \right]^{\frac{1}{q}} \right\} \\
& \text{and} \\
& \left| \frac{1}{3} \ln(G(a, b)) + \frac{2}{3} \ln(A(a, b)) - \ln(I(a, b)) \right| \\
& \leq \frac{(b-a)^2}{48} \left(\frac{2^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)}\right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1}\right)^{i-1} \frac{1}{b^{2q}} + \left(\frac{q+3}{q+1}\right)^{2-i} \frac{1}{a^{2q}} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. The assertions follow from Proposition 1, 2 and 3 applied to convex mapping $f(x) = \ln(x)$, $x \in [a, b]$, respectively. \square

Proposition 6. Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q \geq 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $n \in \mathbb{N}$, $n > 2$ we have

$$\begin{aligned}
& |A^n(a, b) - L_n^n(a, b)| \\
& \leq \frac{n(n-1)(b-a)^2}{16} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)}\right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1}\right)^{i-1} b^{(n-2)q} + \left(\frac{q+3}{q+1}\right)^{2-i} a^{(n-2)q} \right]^{\frac{1}{q}} \right\} \\
& \text{and} \\
& |A(a^n, b^n) - L_n^n(a, b)| \\
& \leq \frac{n(n-1)(b-a)^2}{16} \left(\frac{2^{p+1}-1}{p+1}\right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)}\right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1}\right)^{i-1} b^{(n-2)q} + \left(\frac{q+3}{q+1}\right)^{2-i} a^{(n-2)q} \right]^{\frac{1}{q}} \right\} \\
& \text{and} \\
& \left| \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(a, b) \right| \\
& \leq \frac{n(n-1)(b-a)^2}{48} \left(\frac{2^{p+1}-1}{3(p+1)}\right)^{\frac{1}{p}} \left(\frac{1}{2(q+2)}\right)^{\frac{1}{q}} \left\{ \sum_{i=1}^2 \left[\left(\frac{q+3}{q+1}\right)^{i-1} b^{(n-2)q} + \left(\frac{q+3}{q+1}\right)^{2-i} a^{(n-2)q} \right]^{\frac{1}{q}} \right\}.
\end{aligned}$$

Proof. The assertions follow from Proposition 1, 2 and 3 applied to convex mapping $f(x) = x^n$, $x \in [a, b]$, respectively. \square

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